Group Homology and Tate Cohomology

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1 Group Homology

Let G be a group and A be a G-module. Recall that we defined the n-th cohomology group of G with coefficients in A to be

$$H^n(G, A) := \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

Similarly, define the n-th homology group of G with coefficients in A to be

$$H_n(G, A) := \mathrm{Tor}^{\mathbb{Z}[G]}_n(\mathbb{Z}, A).$$

We will be using the standard resolution (free resolution) of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules:

$$... \to \mathbb{Z}[G^{n+1}] \xrightarrow{d_n} \mathbb{Z}[G^n] \xrightarrow{d_{n-1}} ... \xrightarrow{d_2} \mathbb{Z}[G^2] \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{1}$$

where $d_n(g_0, ..., g_n) := \sum\limits_{i=0}^{n}(-1)^i(g_0, ..., \hat{g}_i, ..., g_n)$ and $\epsilon(\sum a_g g) = \sum a_g$ is the augementation map.

Now, applying the left exact $\mathrm{Hom}_{\mathbb{Z}[G]}(\_, A)$ to 1, we obtain

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \xrightarrow{d^0} \mathrm{Hom}_{\mathbb{Z}[G^2]}(\mathbb{Z}[G], A) \xrightarrow{d^1} \mathrm{Hom}_{\mathbb{Z}[G^3]}(\mathbb{Z}[G], A) \xrightarrow{d^2} ...$$

It follows that for $n \geq 1$,

$$H^n(G, A) = \mathrm{Im}(d^n) \big/ \ker(d^{n-1})$$

and define $H^0(G, A) := \ker(d^0) \cong A^G$.

Similarly, by applying right exact functor $\_ \otimes_{\mathbb{Z}[G]} A$ to 1, we obtain

$$... \to \mathbb{Z}[G^{n+1}] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_n} \mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_{n-1}} ... \xrightarrow{d_2} \mathbb{Z}[G^2] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_1} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_0} 0.$$

Hence, we have

$$H_n(G, A) = \ker(d_n) \big/ \mathrm{Im}(d_{n+1}).$$

Just as the group cohomology, we have

Proposition 1.1. *Let G be a group.*

(a) A G-mod homomorphism $\alpha : A \to B$ induces

$$\begin{aligned}\alpha_{n,*} : H_n(G,A) &\to H_n(G,B)\\ g\otimes a &\mapsto g\otimes \alpha(a).\end{aligned}$$

(b) A short exact sequence of G-mods

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$$

induces a long exact sequence of abelian groups:

$$\ldots \to H_1(G,C) \xrightarrow{\delta} H_0(G,A) \xrightarrow{i_*} H_0(G,B) \xrightarrow{\pi_*} H_0(G,C) \to 0.$$

Recall that the kernel of the augmentation map

$$\begin{aligned}\epsilon : \mathbb{Z}[G] &\to \mathbb{Z}\\ \sum a_g g &\mapsto \sum a_g\end{aligned}$$

is called the augmentation ideal, denoted as I_G. One can easily check that I_G is a free $\mathbb{Z}$-module with basis $\{g-1 : g \in G\}$.

Lemma 1.2. *$H_0(G,A) \cong A/I_GA =: A_G$.*

Proof. We have the sequence

$$\ldots \to \mathbb{Z}[G^2] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_1} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_0} 0.$$

We want to show that $\mathrm{Im}(d_1) \cong I_GA$. Under the identification $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \cong A$ (as G-modules), it's easy to see that $I_GA \in \mathrm{Im}(d_1)$. For a simple tensor $(g_0,g_1)\otimes a \in \mathbb{Z}[G^2]\otimes_{\mathbb{Z}[G]} A$,

$$d_1\big((g_0,g_1)\otimes a\big) = (g_0-g_1)a \in I_GA.$$

Thus, $\mathrm{Im}(d_1) \cong I_GA$. $\square$

Remark 1.3. For a G-module A,

$$\begin{aligned}H^0(G,A) &= A^G \text{ is the largest trivial } G\text{-submodule}\\ H_0(G,A) &= A_G \text{ is the largest trivial } G\text{-module that's a quotient.}\end{aligned}$$

Definition 1.4. Let H be a subgroup of G and A be a H-module. Define

$$\begin{aligned}\mathrm{Ind}_H^G(A) &:= \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A\\ \mathrm{CoInd}_H^G(A) &:= \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], A).\end{aligned}$$

$\mathrm{Ind}_H^G(A)$ and $\mathrm{CoInd}_H^G(A)$ are given the G-module structures: for $g, g' \in G, a \in A$ and $\phi \in \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], A)$,

$$\begin{aligned}g'\cdot(g\otimes a) &:= g'g\otimes a\\ (g'\cdot\phi)(g) &:= \phi(gg').\end{aligned}$$

We say B is an *induced* (resp *coinduced*) G-module if there exists an abelian group A such that

$$B = \mathrm{Ind}_1^G(A) =: \mathrm{Ind}^G(A)$$

(resp $B = \mathrm{CoInd}_1^G(A) =: \mathrm{CoInd}^G(A)$.

Lemma 1.5. *Let G be a group and A be an abelian group. Then,*

$$H_n(G, Ind^G(A)) = \begin{cases} A, & n = 0 \\ 0, & n \geq 1 \end{cases}.$$

$$H^n(G, CoInd^G(A)) = \begin{cases} A, & n = 0 \\ 0, & n \geq 1 \end{cases}.$$

Proof. We prove the statement for homology with coeffients in $\mathrm{Ind}^G(A)$. For $n = 0$, we know that

$$\begin{aligned} H_0(G, \mathrm{Ind}^G(A)) &\cong (\mathrm{Ind}^G(A))_G \\ &= \big(\mathbb{Z}[G] \otimes A\big)_G \\ &= A[G] \Big/ I_G A[G] \\ &\cong A. \end{aligned}$$

Now suppose $n \geq 1$. Then, as G modules, we have

$$\begin{aligned} \mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} \mathrm{Ind}^G(A) &= \mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \\ &= \mathbb{Z}[G^n] \otimes_{\mathbb{Z}} A. \end{aligned}$$

This shows that $H_n(G, \mathrm{Ind}^G(A)) \cong H_n(\{1\}, A)$. To compute this we can consider the free resolution by $\{1\}$-modules (ie abelian groups):

$$0 \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0.$$

This gives $H_n(G, \mathrm{Ind}^G(A)) \cong H_n(\{1\}, A) = 0$. □

Remark 1.6. This result can be thought of as a special case of Shapiro's lemma.

Lemma 1.7. *Suppose H is a subgroup of G with finite index and A is an H-module. Then, the map*

$$\begin{aligned} CoInd_H^G(A) &\xrightarrow{\cong} Ind_H^G(A) \\ \phi &\mapsto \sum_{g \in G/H} g^{-1} \otimes \phi(g) \end{aligned}$$

is a G-module isomorphism.

Proof. One can check that the inverse is given by

$$\begin{aligned} \mathrm{Ind}_H^G(A) &\to \mathrm{CoInd}_H^G(A) \\ g \otimes a &\mapsto \left(g' \mapsto \begin{cases} a, & g' = g \\ 0, & g' \neq g \end{cases}\right). \end{aligned}$$

□

Corollary 1.8. *Let G be a finite group. If B is induced or coinduced by an abelian group A, then*

$$\begin{aligned} H_0(G, B) &= H^0(G, B) = A \\ H_n(G, B) &= H^n(G, B) = 0, \text{ for all } n \geq 1. \end{aligned}$$

2 Tate Cohomology

In this section, we assume G is finite and A is a G-module.

Definition 2.1. Define the norm map

$$\begin{aligned} N_G : A &\to A \\ a &\mapsto \sum_{g \in G} ga. \end{aligned}$$

Remark 2.2. We have

$$\begin{aligned} I_G A &\subseteq \ker(N_G) \\ N_G(A) &\subset A^G. \end{aligned}$$

Thus, the norm map induces

$$\widehat{N}_G : A_G \to A^G.$$

Definition 2.3. (**Tate Cohomology**) Suppose G is finite and A is a G-module. Define the Tate cohomology groups as

$$\widehat{H}^n(G, A) = \begin{cases} H^n(G, A), & n \geq 1 \\ H_{-n-1}(G, A), & n \leq -2 \\ \mathrm{coker}(\widehat{N}_G), & n = 0 \\ \ker(\widehat{N}_G), & n = -1 \end{cases}.$$

Theorem 2.4. *Let G be a finite group and A be a G-module. Then, every short exact sequence of G-modules*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

induces a doubly infinite long exact sequence of Tate cohomology groups

$$\ldots \to \widehat{H}^n(G, A) \xrightarrow{\hat{\alpha}^n} \widehat{H}^n(G, B) \xrightarrow{\hat{\beta}^n} \widehat{H}^n(G, C) \xrightarrow{\hat{\delta}^n} \widehat{H}^{n+1}(G, A) \to \ldots$$

Proof. Consider the diagram:

$$\begin{tikzcd} \ldots \arrow[r] & H_1(G, C) \arrow[r, "\delta_0"] & A_G \arrow[r, "\alpha_0"] \arrow[d, "\widehat{N}_G^A"] & B_G \arrow[r, "\beta_0"] \arrow[d, "\widehat{N}_G^B"] & C_G \arrow[r] \arrow[d, "\widehat{N}_G^C"] & 0 & \\ & 0 \arrow[r] & A^G \arrow[r, "\alpha^0"] & B^G \arrow[r, "\beta^0"] & C^G \arrow[r, "\delta^0"] & H^1(G, A) \arrow[r] & \ldots \end{tikzcd}$$

Note that the first exact sequence is given by the homology groups and second exact sequence is given by cohomology groups. We denote $\widehat{N}_G^A$ as the norm map from $A_G \to A^G$. Similarly for B and C. It's easy to check that this diagram is commutative.

Now by applying snake lemma to the diagram above, we obtain

$$
\begin{tikzcd}
 & & \ker(\widehat{N}_G^A) \arrow[r, "\widehat{\alpha}_0"] \arrow[d, hook] & \ker(\widehat{N}_G^B) \arrow[r, "\widehat{\beta}_0"] \arrow[d, hook] & \ker(\widehat{N}_G^C) \arrow[d, hook] \arrow[dddll, rounded corners, to path={-| ([xshift=1em]\tikztostart.east) |- ([yshift=-0.5em]\tikztostart.south |- \tikztotarget.north) -| ([xshift=-1em]\tikztotarget.west) |- (\tikztotarget)}] & & \\
\dots \arrow[r] & H_1(G,C) \arrow[r, "\delta_0"] \arrow[ur, dashed, "\widehat{\delta}_0"] & A_G \arrow[r, "\alpha_0"] \arrow[d, "\widehat{N}_G^A"] & B_G \arrow[r, "\beta_0"] \arrow[d, "\widehat{N}_G^B"] & C_G \arrow[r] \arrow[d, "\widehat{N}_G^C"] & 0 & \\
 & 0 \arrow[r] & A^G \arrow[r, "\alpha^0"] \arrow[d, two heads] & B^G \arrow[r, "\beta^0"] \arrow[d, two heads] & C^G \arrow[r, "\delta^0"] \arrow[d, two heads] & H^1(G,A) \arrow[r] & \dots \\
 & & \operatorname{coker}(\widehat{N}_G^A) \arrow[r, "\widehat{\alpha}^0"] & \operatorname{coker}(\widehat{N}_G^B) \arrow[r, "\widehat{\beta}^0"] & \operatorname{coker}(\widehat{N}_G^C) \arrow[ur, dashed, "\widehat{\delta}^0"] & &
\end{tikzcd}
$$

By snake lemma and the definition of Tate cohomology groups, it remains to show the following:

(i) δ_0 factors through $\ker(\widehat{N}_G^A)$ and $\ker(\widehat{\alpha}_0) = \operatorname{Im}(\widehat{\delta}_0)$.

(ii) δ^0 factors through $\operatorname{coker}(\widehat{N}_G^C)$ and $\ker(\widehat{\delta}^0) = \operatorname{Im}(\widehat{\beta}^0)$.

For (i), since $\operatorname{Im}(\delta_0) = \ker(\alpha_0)$ and $\ker(\widehat{\alpha}_0) \subset \ker(\widehat{N}_G^A)$, it suffices to show $\ker(\alpha_0) = \ker(\widehat{\alpha}_0)$. It is easy to see one inclusion $\ker(\widehat{\alpha}_0) \subset \ker(\alpha_0)$. For the inclusion $\ker(\alpha_0) \subset \ker(\widehat{\alpha}_0)$, let $a \in A_G$ such that $\alpha_0(a) = 0$. then since α^0 is injective, we have $\widehat{N}_G^A(a) = 0$, i.e. $a \in \ker(N_G^A)$ and so $a \in \ker(\widehat{\alpha}_0)$.

For (ii), to show δ^0 factors through $\operatorname{coker}(\widehat{N}_G^C)$, we need to show $\operatorname{Im}(N_G^C) \subset \ker(\delta^0)$. Since $\ker(\delta^0) = \operatorname{Im}(\beta^0)$, this is equivalent to showing $\operatorname{Im}(N_G^C) \subset \operatorname{Im}(\beta^0)$. Let $c \in \operatorname{Im}(N_G^C)$. Then, there exists $c' \in C_G$ such that $\widehat{N}_G^C(c') = c$. As β_0 is surjective, we get an element $b \in B^G$ with $\beta^0(b) = c$, i.e. $c \in \operatorname{Im}(\beta^0)$. Now because the diagram is commutative, we also have $\ker(\widehat{\delta}^0) = \operatorname{Im}(\widehat{\beta}^0)$. $\square$

We have shown before that for finite group G, induced and coinduced G-modules have trivial cohomology in positive degrees. The following proposition shows that the definition of Tate cohomology is the minimal modification so that this is correct for all integer degrees.

Proposition 2.5. *Let G be a finite group. Suppose B is induced or coinduced. Then,*

$$\widehat{H}^n(G,B) = 0$$

for all $n \in \mathbb{Z}$.

Proof. It suffices to verify $\widehat{H}^0(G,B)$ and $\widehat{H}^{-1}(G,B)$ for $B = \operatorname{Ind}^G(A)$ where A is an abelian group. By definition,

$$\widehat{H}^0(G,B) = B^G \Big/ \widehat{N}_G^G$$
$$\widehat{H}^{-1}(G,B) = \ker(\widehat{N}_G)$$

where $\widehat{N}_G : B_G \to B^G$ is induced by

$$N_G : B \to B.$$

Now, since $B = \mathrm{Ind}^G(A) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ and the G-action is only applied to $\mathbb{Z}[G]$, we have

$$\begin{aligned} &\ker(N_G) = I_G B \\ &\mathrm{Im}(N_G) = N_G \mathbb{Z} \otimes_{\mathbb{Z}} A = \mathbb{Z}[G]^G \otimes_{\mathbb{Z}} A = B^G. \end{aligned}$$

Hence, $\widehat{H}^{-1}(G, B)$ and $\widehat{H}^0(G, B)$ are both 0. $\square$

Example 2.6. We show that for a finite group G,

$$\widehat{H}^{-2}(G, \mathbb{Z}) := H_1(G, \mathbb{Z}) = G^{ab}$$

where G^{ab} is the abelianization of G.

Consider the exact sequence of G-modules:

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Note that since $\mathbb{Z}[G]$ is obviously free viewed as a $\mathbb{Z}[G]$-module, $\widehat{H}^n(G, \mathbb{Z}) = 0$ for all $n \in \mathbb{Z}$. In particular, $H_1(G, \mathbb{Z}) = 0$. So we have the exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(G,\mathbb{Z}) & \xrightarrow{\delta} & H_0(G, I_G) & \xrightarrow{0} & H_0(G, \mathbb{Z}[G]) & \longrightarrow & H_0(\mathbb{Z}[G], \mathbb{Z}) & \longrightarrow & 0. \\ & & & & \| & & \| & & \| & & \\ & & & & I_G/I_G^2 & & \mathbb{Z}[G]/I_G & & \mathbb{Z} & & \end{array}$$

The map $H_0(G, I_G) \to H_0(G, \mathbb{Z}[G])$ is induced by inclusion $I_G \hookrightarrow \mathbb{Z}[G]$, and hence is the zero map. Now we have

$$H_1(G, \mathbb{Z}) \underset{\cong}{\xrightarrow{\delta}} I_G/I_G^2.$$

Note that this shows that I_G/I_G^2 is an abelian group. On the other hand, consider the map

$$\begin{aligned} \gamma : I_G/I_G^2 &\to G^{ab} \\ (g-1) + I_G^2 &\mapsto g. \end{aligned}$$

We leave to the readers to check this is a well-defined group isomorphism. It follows that

$$H_1(G, \mathbb{Z}) \underset{\cong}{\xrightarrow{\delta}} I_G/I_G^2 \underset{\cong}{\xrightarrow{\gamma}} G^{ab}.$$

3 Tate cohomology of Cyclic groups & Herbrand Quotients

In this section, we assume $G = \langle\sigma\rangle$ is a finite cyclic group. So, the augmentation ideal $I_G = (\sigma - 1)$ is an ideal in $\mathbb{Z}[G]$. We will also make use of the free resolution of $\mathbb{Z}$ by G-modules

$$...\mathbb{Z}[G] \xrightarrow{N_G} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0. \tag{2}$$

Theorem 3.1. *Suppose G is a finite cyclic group and A is a G-module. Then,*

$$\widehat{H}^n(G, A) = \widehat{H}^{n+2}(G, A)$$

for all $n \in \mathbb{Z}$.

Proof. We first apply the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\_, A)$ to 3,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \arrow[r, "(\sigma-1)^*"] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \arrow[r, "(N_G)^*"] \arrow[d, equal] & ... \\
0 \arrow[r] & A \arrow[r, "\sigma-1"] & A \arrow[r, "N_G"] & ...
\end{tikzcd}$$

So for $n \geq 1$,

$$\begin{cases} H^{2n}(G, A) & = \ker(\sigma - 1)/N_G(A) = A^G/\widehat{N}_G(A_G) =: \widehat{H}^0(G, A) \\ H^{2n-1}(G, A) & = \ker(N_G)/(\sigma - 1)A. \end{cases}$$

On the other hand, we apply $\_ \otimes_{\mathbb{Z}[G]} A$ to and get

$$\begin{tikzcd}
... \arrow[r, "(\sigma-1)\otimes\mathrm{id}"] & \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \arrow[r, "N_G\otimes\mathrm{id}"] \arrow[d, equal] & \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \arrow[r, "(\sigma-1)\otimes\mathrm{id}"] \arrow[d, equal] & \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \arrow[r] \arrow[d, equal] & 0 \\
... \arrow[r, "\sigma-1"] & A \arrow[r, "N_G"] & A \arrow[r, "\sigma-1"] & A \arrow[r] & 0
\end{tikzcd}.$$

So for $n \geq 1$,

$$\begin{cases} H_{2n}(G, A) & = \ker(N_G)/(\sigma - 1)A = \ker(\widehat{N}_G) =: \widehat{H}^{-1}(G, A) \\ H_{2n-1}(G, A) & = \ker(\sigma - 1)/N_G(A). \end{cases}$$

$\square$

Now that we know the cohomology groups of cyclic group is periodic with period 2, we can define the following:

Definition 3.2. Suppose G is a finite cyclic group and A is a G-module. The *Herbrand quotient* of A is

$$h(A) := \frac{|\widehat{H}^0(G, A)}{|\widehat{H}^{-1}(G, A)|}$$

whenever both factors are finite.

Corollary 3.3. *Let G be a finite cyclic group and consider the short exact sequence*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$

Then we have an exact hexagon

$$\begin{tikzcd}
 & \widehat{H}^0(G,A) \arrow[r, "\widehat{\alpha}^0"] & \widehat{H}^0(G,B) \arrow[dr, "\widehat{\beta}^0"] & \\
\widehat{H}^{-1}(G,C) \arrow[ur, "\widehat{\delta}^{-1}"] & & & \widehat{H}^0(G,C) \arrow[dl, "\delta^0"] \\
 & \widehat{H}^{-1}(G,B) \arrow[ul, "\widehat{\beta}^{-1}"] & \widehat{H}^{-1}(G,B) \arrow[l, "\widehat{\alpha}^{-1}"'] &
\end{tikzcd} .$$

Corollary 3.4. *Suppose G is a finite cyclic group and consider the short exact sequence*

$$0 \to A \to B \to C \to 0.$$

If any two of $h(A), h(B), h(C)$ are defined, so is the third and

$$h(B) = h(A)h(C)$$

Proof. Apply the exact hexagon. □

Corollary 3.5. *Let G be a finite cyclic group. Then,*

$$h(A \oplus B) = h(A)h(B).$$

Lemma 3.6. *Let G be a finite cyclic group and A be induced, coinduced or finite G-module. Then, $h(A) = 1$.*

Proof. If A is induced or coinduced, then all Tate cohomology groups are trivial.

Suppose A is finite. Consider the short exact sequence

$$0 \to A^G \xrightarrow{i} A \xrightarrow{\sigma - 1} A \xrightarrow{\pi} A_G \to 0.$$

Then,

$$|A^G| = |\ker(\sigma - 1)| = |A_G|.$$

On the other hand, from $\widehat{N}_G : A_G \to A^G$, we have

$$\begin{aligned}
|\ker(\widehat{N}_G)| &= \frac{|A_G|}{|\widehat{N}_G(A_G)|} \\
&= \frac{|A^G|}{|\widehat{N}_G(A_G)|} \\
&= \left| A^G / \widehat{N}_G(A_G) \right| \\
&= |\mathrm{coker}(\widehat{N}_G)|.
\end{aligned}$$

Thus, $h(A) = 1$. □

Corollary 3.7. *Let G be a finite cyclic group. and A be a G-module which is also finitely generated as an abelian group. Then,*

$$h(A) = h(A/A_{tor}).$$

Proof. Consider the short exact sequence

$$0 \to A_{tor} \to A \to A/A_{tor} \to 0$$

and apply the lemma to the finite group A_{tor}. □

Corollary 3.8. *Let G be a finite cyclic group and A be a trivial G-module which is also finitely generated as an abelian group of rank r. Then,*

$$h(A) = |G|^r.$$

Corollary 3.9. *Let G be a finite cyclic group. Suppose*

$$\alpha : A \to B$$

is a G-module homomorphism that has finite kernel and finite cokernel. Then, $h(A) = h(B)$.

Proof. Consider two exact sequences:

$$0 \to \ker(\alpha) \to A \to \mathrm{Im}(\alpha) \to 0$$
$$0 \to \mathrm{Im}(\alpha) \to B \to \mathrm{coker}(\alpha) \to 0.$$

Then, $h(\ker(\alpha) = 1 = \mathrm{coker}(\alpha)$ and

$$\begin{aligned} h(A) &= h(\ker(\alpha))h(\mathrm{Im}(\alpha)) \\ &= h(\mathrm{Im}(\alpha)) \\ &= h(B). \end{aligned}$$

□

Corollary 3.10. *Let G be a finite cyclic group and A be a G-module containing a G-submodule B of finite index. Then,*

$$h(A) = h(B).$$

Proof. Apply the lemma to the inclusion map $B \hookrightarrow A$. □

4 Tate's Theorem

We first make a remark on induced and coinduced modules. Suppose A is a G-module and $\mathring{A}$ be the underlying abelian group. Recall that $\mathrm{Ind}^G(\mathring{A})$ and $\mathrm{CoInd}^G(\mathring{A})$ have the following G-module structure: for $g, z \in G, a \in \mathring{A}$ and $\phi \in \mathrm{CoInd}^G(\mathring{A})$,

$$\begin{aligned} g \cdot (z \otimes a) &= gz \otimes a \\ (g \cdot \phi)(z) &= \phi(zg). \end{aligned}$$

We give $\mathrm{Ind}^G(A)$ and $\mathrm{CoInd}^G(A)$ the following G-module structures: for $g, z \in G, a \in A$ and $\phi \in \mathrm{CoInd}^G(A)$,

$$\begin{aligned} g \cdot (z \otimes a) &= gz \otimes ga \\ (g \cdot \phi)(z) &= g(\phi(g^{-1}z)). \end{aligned}$$

The following proposition tells us that they are all isomorphic as G-modules.

Proposition 4.1. *Suppose G is a group (not necessarily finite) and A is a G-module. Then,*

$$\begin{aligned} Ind^G(A) &\to Ind^G(\mathring{A}) \\ g \otimes a &\mapsto g \otimes ga \end{aligned}$$

and

$$\begin{aligned} CoInd^G(A) &\to CoInd^G(\mathring{A}) \\ \phi &\mapsto (\psi : z \mapsto z\phi(z^{-1}) \end{aligned}$$

are G-module isomorphisms. In particular, when G is finite,

$$Ind^G(\mathring{A}) \cong Ind^G(A) \cong CoInd^G(A) \cong CoInd^G(\mathring{A}).$$

In the remaining section, we assume G is a finite group. We would either let A be an abelian group or a G-module. Consider the exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

By applying $\_ \otimes_{\mathbb{Z}} A$, we get an exact sequence

$$0 \to I_G \otimes_{\mathbb{Z}} A \to \mathrm{Ind}^G(A) \to A \to 0.$$

By applying $\mathrm{Hom}_{\mathbb{Z}}(\_, A)$, we get an exact sequence

$$0 \to A \to \mathrm{CoInd}^G(A) \to \mathrm{Hom}_{\mathbb{Z}}(I_G, A) \to 0.$$

For a proof of these two facts, refer to [1].

Theorem 4.2. ***(Dimension Shifting)*** *Let G be finite and A be a G-module. For a subgroup H in G, we have*

$$\begin{aligned} \widehat{H}^n(H, A) &\cong \widehat{H}^{n-1}(H, Hom_{\mathbb{Z}}(I_G, A)) \\ \widehat{H}^n(H, A) &\cong \widehat{H}^{n+1}(H, I_G \otimes_{\mathbb{Z}} A). \end{aligned}$$

The theorem tells us that Tate cohomology groups are completely determined if we knew one cohomology group. Before stating Tate's theorem, we need another criterion for cohomological triviality.

Theorem 4.3. *Let G be a finite group and A be a G-module. If $H^1(H, A) = H^2(H, A) = 0$ for all $H \leq G$, then*

$$\widehat{H}(G, A) = 0 \textit{ for all } n \in \mathbb{Z}.$$

Proof. This is proved step-by-step. First of all, this is obvious for cyclic groups. One can then apply inflation-restriction sequence with dimension shifting to prove for solvable group. Lastly for arbitrary finite group G, one uses the composite of corestriction and restriction on Sylow p-subgroups. □

Theorem 4.4. ***(Tate's Theorem)*** *Let G be a finite group and A be a G-module. Suppose for all subgroups $H \leq G$, we have*

(a) $H^1(H, A) = 0$.

(b) $H^2(H, A)$ *is cyclic of order* $|H|$.

Then, for a generator $\varphi \in H^2(G, A)$, there exists an isomorphism

$$\Phi_\varphi^n : \widehat{H}^n(G, \mathbb{Z}) \xrightarrow{\cong} \widehat{H}^{n+2}(G, A)$$

which only depends on the choice of φ.

Proof. Fix a generator $\varphi : G^2 \to A$ of $H^2(G, A)$. Define

$$A(\varphi) := A \oplus \text{ free abelian group with basis } \{x_g : g \in G \setminus \{1\}\}.$$

The G-action on A is given by

$$g \cdot x_h := x_{hg} - x_g + \phi(g, h)$$

with $x_1 := \phi(1, 1)$. By using the fact that φ is a cocycle, one can check that this gives $A(\varphi)$ a G-module structure.

Let $i : A \hookrightarrow A(\varphi)$ be the inclusion map. Then, note that $i \circ \varphi : G^2 \to A(\varphi)$ is also a cocycle. Define a 1-cocyle

$$\begin{aligned} \chi : G &\to A(\varphi) \\ g &\mapsto x_g. \end{aligned}$$

We claim that $d(\chi) = i \circ \varphi$. In fact for $g_1, g_2 \in G$,

$$\begin{aligned} d(\chi)(g_1, g_2) &= g_2 \cdot \chi(g_1) - \chi(g_1 g_2) + \chi(g_2) \\ &= g_2 \cdot x_{g_1} - x_{g_1 g_2} + x_{g_2} \\ &= x_{g_1 g_2} - x_{g_2} + \phi(g_1, g_2) \\ &= \phi(g_1, g_2). \end{aligned}$$

This shows that $i \circ \varphi$ is a 2-coboundary. Since $H^2(G, A)$ is generated by φ, the map

$$\begin{aligned} i^2 : H^2(G, A) &\to H^2(G, A(\varphi)) \\ [\varphi] &\mapsto [i \circ \varphi] = 0 \end{aligned}$$

induced by the inclusion $i : A \hookrightarrow A(\varphi)$ is the zero map.

On the other hand, define

$$\begin{aligned} \phi : A(\varphi) &\to I_G \\ x_g &\mapsto g - 1 \\ a &\mapsto 0 \end{aligned}$$

for any $a \in A$. Now for every subgroup $H \leq G$, we have the following two exact sequences of H-modules,

$$0 \to A \xrightarrow{i} A(\varphi) \xrightarrow{\phi} I_G \to 0 \tag{3}$$

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0. \tag{4}$$

The sequence 3 induces a long exact sequence

$$\begin{tikzcd} H^1(H, A) \arrow[r, "i^1"] & H^1(H, A(\varphi)) \arrow[r, "\phi^1"] & H^1(H, I_G) \arrow[dll] \\ H^2(H, A) \arrow[r, "i^2=0"] & H^2(H, A(\varphi)) \arrow[r, "\phi^2"] & H^2(H, I_G) \end{tikzcd}.$$

By assumption, $H^1(H, A) = 0$ and $H^2(H, A) = \mathbb{Z}/|H|\mathbb{Z}$ for every $H \leq G$. Also, from sequence and the fact that $\widehat{H}^n(G, \mathbb{Z}[G]) = 0$, we know that

$$\begin{aligned} H^1(H, A) &= \widehat{H}^0(H, \mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z} \\ H^2(H, I_G) &= H^1(H, \mathbb{Z}) = \mathrm{Hom}(H, \mathbb{Z}) = 0. \end{aligned}$$

From these, we deduce that $H^1(H, A(\varphi)) = H^2(H, A(\varphi)) = 0$ for all subgroups $H \leq G$. It follows from previous theorem that

$$\widehat{H}^n(G, A(\varphi)) = 0$$

for all $n \in \mathbb{Z}$. We then define

$$\Phi^n_\varphi : \widehat{H}^n(G, \mathbb{Z}) \xrightarrow{\delta^n} \widehat{H}^{n+1}(G, I_G) \xrightarrow{\delta^{n+1}_\varphi} \widehat{H}^{n+2}(G, A)$$

where δ^n is the connecting map in the long exact sequence for sequence 4 and δ^{n+1}_φ is the connecting homomorphism in the long exact sequence for sequence 3. Since $\widehat{H}^n(G, A(\varphi)) = 0 = \widehat{H}^n(G, \mathbb{Z}[G])$, these two maps are isomorphism. Hence the composite Φ^n_φ is an isomorphism, which concludes the proof. □

References

[1] Romyar Sharifi, *Group and Galois Cohomology.* `http://math.ucla.edu/~sharifi/lecnotes.html`.